Bilangan: Jurnal Ilmiah Matematika, Kebumian dan Angkasa
Volume. 3 Nomor. 6, Desember 2025
e-ISSN : 3032-7113; p-ISSN : 3032-6389, Hal. 01-10
DOI: https://doi.org/10.62383/bilangan.v3i6.849
Tersedia: https://journal.arimsi.or.id/index.php/Bilangan

Uncertainty Principle for the Hartley Transform: Direct and Fourier–Based Approaches

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Abstract. The Hartley transform provides a real-valued alternative to the classical Fourier transform, offering structural advantages for the analysis of real-valued signals. This paper presents a systematic study of the continuous Hartley transform, including its definition, inversion formula, Plancherel identity, and core operational properties such as shifting, modulation, and convolution. The analytical framework is developed in parallel with the classical Fourier theory to highlight structural similarities and distinctions between the two transforms. Furthermore, we establish a Hartley-type Heisenberg uncertainty principle using two complementary approaches: a direct method based on intrinsic properties of the Hartley kernel, and a Fourier-based method that exploits the algebraic relationship between the Hartley and Fourier transforms. These results provide a unified and rigorous foundation for understanding uncertainty relations within real-valued transform frameworks, and they demonstrate the continued relevance of the Hartley transform in harmonic analysis, integral transforms, and modern signal processing.

Keywords: Fourier transform; Hartley transform; Heisenberg inequality; Signal Analysis; Uncertainty principle.

1. INTRODUCTION

The Fourier transform is one of the most fundamental tools in mathematical analysis, signal processing, and communication theory. Its ability to decompose signals into frequency components has led to powerful analytic methods used across science and engineering (Stein & Shakarchi, 2003; Folland, 2009). Despite its widespread utility, the Fourier transform is inherently complex-valued, which may be unnecessary or computationally inefficient in applications where the underlying data are entirely real.

To address this issue, Hartley introduced in 1942 a fully real-valued analogue of the Fourier transform (Hartley, 1942). The Hartley transform employs the real kernel $cas(x) = \cos x + \sin x$, producing a transform that is self-inverse and avoids complex arithmetic. Its theoretical foundations and computational significance were later solidified through Bracewell's modern treatment (Bracewell, 1986). Subsequent work extended Hartley's framework to multidimensional, generalized, and fast computational settings (Lohmann et al., 1989; Bracewell, 1984; Hargreaves, 1991).

The discrete Hartley transform (DHT), first introduced in Bracewell (1983), broadened the transform's impact in digital signal processing, enabling efficient real-valued convolution, filtering, and fast algorithmic implementations (Feldman, 1999; McLaren & Smith, 1998; Martucci, 2015). Additional studies have highlighted its advantages in numerical integration, image processing, and real-valued filter design (Vlček & Novák, 1999; Zadeh & Reibman, 2002; Bose & Boo, 2005).

Naskah Masuk: 29 Agustus 2025; Revisi: 30 September 2025; Diterima: 28 Oktober 2025; Terbit: 03 Desember 2025;

Beyond computational considerations, several authors have emphasized the structural and functional-analytic relationships between the Hartley and Fourier transforms, including equivalence of energy identities, symmetry properties, and harmonic-analytic behavior (Oppenheim & Willsky, 1997). More advanced works have connected Hartley-type transforms to generalized uncertainty principles and real-valued harmonic analysis (Cowling & Price, 1984; Goh & Pfander, 1993), providing improved understanding of localization and transform-domain constraints.

The purpose of this paper is to provide a rigorous and coherent presentation of the continuous Hartley transform and its key analytic properties. Section 2 introduces the necessary functional-analytic preliminaries, including Lebesgue spaces and the Fourier transform. Section 3 develops the Hartley transform, its inversion formula, Plancherel identity, and core operational properties. Section 4 establishes a Hartley version of the Heisenberg uncertainty principle using two approaches: a direct analytic proof and a Fourier-based method, emphasizing the structural parallels and distinctions between the two transform frameworks.

2. PRELIMINARIES

In this section we recall several basic definitions and notations used throughout the paper.

For $1 \leq r \leq \infty$, the Lebesgue space $L^r(\mathbb{R})$ consists of all measurable functions on $\mathbb{R}$ whose L^r-norm is finite.

Definition 2.1 (The $L^r(\mathbb{R})$Space)

The space $L^r(\mathbb{R})$is defined as

$$\|f\|_{L^r(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(t)|^r \, dt\right)^{\frac{1}{r}}, \qquad 1 \leq r < \infty, \tag{2.1}$$

and for $r = \infty$,

$$\|f\|_{L^\infty(\mathbb{R})} = \operatorname*{ess\,sup}_{t\in\mathbb{R}} |f(t)|. \tag{2.2}$$

The space $L^2(\mathbb{R})$is a Hilbert space with inner product

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(t)\, \overline{g(t)}\, dt. \tag{2.3}$$

We recall the definition of the Fourier transform, which will be used extensively in later sections.

Definition 2.2 (Fourier Transform)

For a function $f \in L^1(\mathbb{R})$, the Fourier transform is defined by

$$\mathcal{F}\{f\}(w) = F(w) = \int_{-\infty}^{\infty} f(t)\ e^{-jwt}\, dt. \tag{2.4}$$

Lemma 2.3 (Inverse Fourier Transform)

If $F = \mathcal{F}\{f\} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} F(w)\ e^{jwt}\, dw.$$

Using the identity $e^{-jwt} = \cos(wt) - j\sin(wt)$, the Fourier transform decomposes as

$$F(w) = F_R(w) + jF_I(w). \tag{2.5}$$

where

$$F_R(w) = \int_{\mathbb{R}} f(t) \cos(wt)\, dt,$$

and

$$F_I(w) = -\int_{\mathbb{R}} f(t) \sin(wt)\, dt.$$

Lemma 2.4 (Parseval's Identity)

For all $f, g \in L^2(\mathbb{R})$, the following identity holds:

$$\int_{\mathbb{R}} f(t)\ \overline{g(t)}\, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}\{f\}(w)\ \overline{\mathcal{F}\{g\}(w)}\, dw.$$

In particular,

$$\|f\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \|\mathcal{F}\{f\}\|_{L^2(\mathbb{R})}^2.$$

Lemma 2.5 (Cauchy–Schwarz Inequality)

For $u, v \in L^2(\mathbb{R})$,

$$\left| \int_{-\infty}^{\infty} u(t)\ v(t)\ dt \right|^2 \leq \left(\int_{-\infty}^{\infty} |u(t)|^2\ dt \right) \left(\int_{-\infty}^{\infty} |v(t)|^2\, dt \right).$$

A standard reference for this inequality is (Stein & Shakarchi, 2003).

3. HARTLEY TRANSFORM AND ITS PROPERTIES

The Hartley transform serves as a real-valued analogue of the classical Fourier transform and provides a convenient framework for the analysis of real signals. This section establishes the basic definition of the transform and develops several fundamental properties that form the analytical foundation for later results, including the inversion formula, Plancherel identity, and operational rules.

Definition 3.1 (Hartley Transform)

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The Hartley transform of f is defined by

$$H(w) = \mathcal{H}\{f\}(w) = \int_{-\infty}^{\infty} f(t)\, cas\,(wt)dt, \tag{3.6}$$

where the cas is given by

$$cas\,(x) = \cos(x) + \sin(x).$$

Example 1. Consider the Gaussian function

$$f(t) = e^{at^2}, \qquad a > 0,$$

compute its Hartley transform

Solution. The Hartley transform of $f(t)$ is defined by

$$\begin{aligned} H(w) = \mathcal{H}\{f\}(w) &= \int_{-\infty}^{\infty} f(t)\, cas(wt)dt \\ &= \int_{-\infty}^{\infty} e^{at^2}\,(\cos(wt) + \sin(wt))dt \end{aligned} \tag{3.7}$$

By separating the integral into cosine and sine components, equation (3.7) becomes

$$H(w) = \underbrace{\int_{-\infty}^{\infty} e^{at^2} \cos(wt)\, dt}_{I_1} + \underbrace{\int_{-\infty}^{\infty} e^{at^2} \sin(wt)\, dt.}_{I_2} \tag{3.8}$$

To evaluate the cosine part I_1, one can complete the square in the exponent, leading to

$$I_1 = \int_{-\infty}^{\infty} e^{at^2} \cos(wt)\, dt = \sqrt{\frac{\pi}{a}}\, e^{-w^2/(4a)}, \tag{3.9}$$

where this uses the standard Gaussian integral

$$\int_{-\infty}^{\infty} e^{a(t-iw/(2a))^2}\, dt = \sqrt{\frac{\pi}{2}}. \tag{3.10}$$

The sine part I_2 of equation (3.8) vanishes due to symmetry, because e^{at^2} is even and $\sin(wt)$ is odd:

$$I_2 = \int_{-\infty}^{\infty} e^{-at^2} \sin(wt)\, dt = 0. \tag{3.11}$$

By substituting equations (3.9) and (3.11) into (3.8), the Hartley transform of the Gaussian function becomes

$$H(w) = \sqrt{\frac{\pi}{2}}\, e^{-w^2/(4a)}. \tag{3.12}$$

Thus, the Hartley transform preserves the Gaussian shape, analogous to the Fourier transform, which illustrates one of the convenient properties of the Hartley transform in signal analysis.

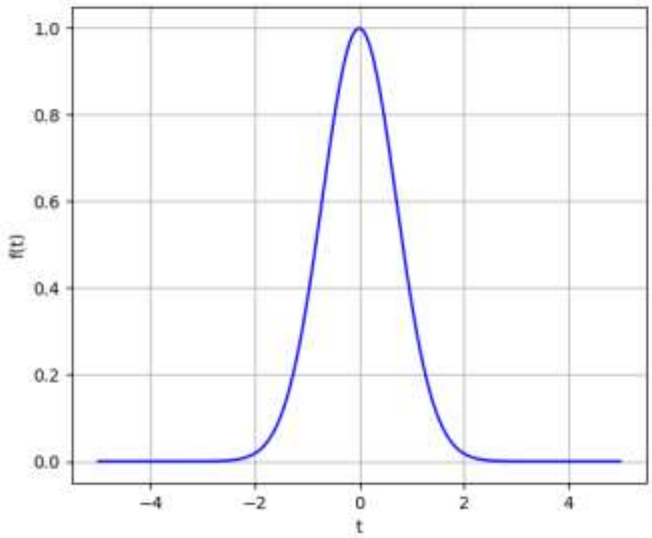

Figure 1. Gaussian function.

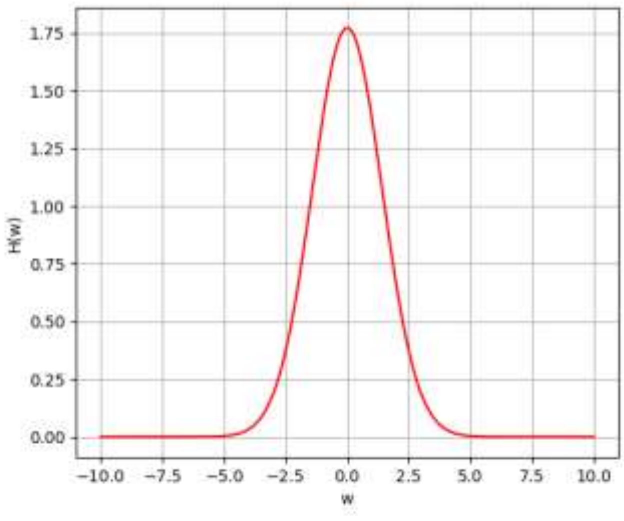

Figure 2. Hartley transform of the Guassian function.

Figure 1 illustrates the graph of the Gaussian function which exhibits a bell–shaped curve symmetric about the vertical axis. The function attains its maximum at $t = 0$ and decays exponentially as t moves away from the center. This plot highlights the strong time-domain localization characteristic of the Gaussian.

Figure 2 shows the graph of the Hartley transform of the Gaussian function. The resulting curve remains smooth, symmetric, and well-localized, reflecting the fact that the Gaussian is preserved (up to scaling factors) under various integral transforms, including the

Hartley transform. This frequency-domain graph illustrates how the energy of the original signal is distributed with respect to the variable w.

Combining equation (2.5) with the expression for $H(w)$ yields the fundamental identity

$$H(w) = F_R(w) - F_I(w), \tag{3.13}$$

which expresses the Hartley transform as a real linear combination of the real and imaginary parts of the Fourier transform.

Theorem 3.1 (Invers Formula)

Let $f \in L^1 \cap L^2$. If $H = \mathcal{H}\{f\}$, then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(w)\, cas\,(tw) dw. \tag{3.14}$$

A full proof may be found in (Bracewell, 1986).

Theorem 3.2 (Plancherel Identity)

For $f \in L^2(\mathbb{R})$, the Hartley transform satisfies

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(w)|^2\, dw. \tag{3.15}$$

The proof follows from the self-inverse property of the Hartley transform; see (Bracewell, 1986).

For convenience, we introduce the notation

$$J(w) = F_R(w) + F_I(w),$$

so that by (3.13),

$$H(w) = F_R(w) - F_I\ \ (w),$$

and

$$J(w) = F_R(w) + F_I(w).$$

Theorem 3.3 (Time-shift Identity)

For $f \in L^1 \cap L^2$ and $a \in \mathbb{R}$,

$$\mathcal{H}\{f(t-a)\}(w) = \cos(aw)H(w) + \sin(aw)J(w). \tag{3.16}$$

Proof. Using the Fourier shift rule,

$$\mathcal{F}\{f(t-a)\}(w) = e^{-aw}F(w),$$

and writing $e^{-aw} = \cos(aw) - j\sin(aw)$ with the decomposition (2.5), we obtain

$$\Re\left(e^{-iaw}F\right) = \cos(aw)\, F_R + \sin(aw)\, F_I,$$

amd

$$\Im\left(e^{-iaw}F\right) = \cos(aw)\, F_I - \sin(aw)\, F_R.$$

Since $\mathcal{H}(g) = \Re(\mathcal{F}g) - \Im(\mathcal{F}g)$ by (3.13), the identity follows.

Theorem 3.4 (Modulation Identity)

For $f \in L^1 \cap L^2$ and $a \in \mathbb{R}$,

$$\mathcal{H}\{e^{iat} f(t)\}(w) = \Re\{F(w-a)\} - \Im\{F(w-a)\}. \tag{3.17}$$

Proof. Using the modulation rule for Fourier transform,

$$\mathcal{F}\{e^{iat} f(t)\}(w) = F(w-a),$$

and applying $\mathcal{H} = \Re\mathcal{F} - \Im\mathcal{F}$, the identity follows.

Theorem 3.5 (Convolution Identity)

For $f, g \in L^1 \cap L^2$,

$$\mathcal{H}\{f * g\}(w) = \Re\{F(w)G(w)\} - \Im\{F(w)G(w)\}. \tag{3.18}$$

Proof. The Fourier convolution rule gives $\mathcal{F}\{f * g\} = F \cdot G$. Applying $\mathcal{H} = \Re\mathcal{F} - \Im\mathcal{F}$ yields the result.

4. HEISENBERG UNCERTAINTY PRINCIPLE FOR THE HARTLEY TRANSFORM

This section establishes an analogue of the classical Heisenberg uncertainty principle in the setting of the Hartley transform. The discussion develops a frequency–time inequality consistent with the Fourier case, but expressed entirely in terms of the real-valued Hartley kernel. Central to the analysis are the time and frequency variances associated with a function and its Hartley transform.

Let

$$H(w) = \mathcal{H}\{f\}(w),$$

denote the Hartley transform of $f \in L^2(\mathbb{R})$. Define the time and frequency variances

$$\sigma_t^2 = \int t^2 \, |f(t)|^2 dt,$$

and

$$\sigma_w^2 = \int w^2 \, |H(w)|^2 dw.$$

Theorem 4.1 (Heisenberg Uncertainty Principle: Direct Proof)

For every $f \in L^2(\mathbb{R})$,

$$\sigma_t^2 \sigma_w^2 \geq \frac{1}{4}\left(\int |f(t)|^2 \, dt\right)^2. \tag{4.19}$$

Proof. Using the Plancherel identity (3.15), we obtain

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(w)|^2 dw. \tag{4.20}$$

Next, consider the integral

$$I := \int_{-\infty}^{\infty} t f(t) \overline{f'(t)} dt. \tag{4.21}$$

By integration by parts (and using the fact that the boundary terms vanish for $f \in L^2(\mathbb{R})$), (4.21) yields

$$I = -\frac{1}{2} \int_{-\infty}^{\infty} |f(t)|^2 dt. \tag{4.22}$$

The derivative of the Hartley kernel is given by

$$\frac{d}{dt} cas(wt) = w(\cos(wt) - \sin(wt)) = w\, cas\left(wt - \frac{\pi}{2}\right). \tag{4.23}$$

To estimate the moments, apply Lemma 2.3 with

$$u(t) = tf(t),$$

and

$$v(t) = cas\,(wt),$$

which leads—after transforming one factor into the frequency domain using (3.15) to the inequality

$$\int_{-\infty}^{\infty} w^2 |H(w)|^2 dw \geq \frac{1}{4} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2. \tag{4.24}$$

Finally, multiplying (4.24) by

$$\sigma_t^2 = \int t^2 \, |f(t)|^2 dt,$$

we obtain the desired uncertainty inequality

$$\sigma_t^2 \sigma_w^2 \geq \frac{1}{4} \left(\int |f(t)|^2 \, dt \right)^2,$$

which completed the proof.

Theorem 4.2 (Heisenberg Uncertainty Principle: Fourier Relation)

For every $f \in L^2(\mathbb{R})$,

$$\sigma_t^2 \sigma_w^2 \geq \frac{1}{4} \left(\int |f(t)|^2 \, dt \right)^2.$$

Proof. We begin by recalling that the Fourier transform of f can be written as

$$|F(w)|^2 = |F_R(w)|^2 + |F_I(w)|^2, \tag{4.25}$$

where F_R and F_I denote the real and imaginary parts of F, respectively.

Moreover, the Hartley transform satisfies

$$H(w) = F_R(w) - F_I(w). \tag{4.26}$$

Substituting (4.26) into (4.27) gives

$$|H(w)|^2 = |F_R(w)|^2 + |F_I(w)|^2 - 2F_R(w)F_I(w). \quad (4.27)$$

Integrating (4.26) over ℝ, and observing that the mixed term integrates to zero by symmetry, yields

$$\int_{\mathbb{R}} |H(w)|^2\, dw = \int_{\mathbb{R}} |F(w)|^2\, dw. \quad (4.28)$$

Using (4.29), the Plancherel identity for the Hartley transform follows:

$$\int_{\mathbb{R}} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} |H(w)|^2\, dw = \frac{1}{2\pi}\int_{\mathbb{R}} |F(w)|^2\, dw. \quad (4.29)$$

From (4.30), it follows that the Hartley and Fourier transforms preserve energy in the same way. Using this equivalence, the classical Fourier Heisenberg inequality (see: Bracewell,1986}) becomes applicable. Thus,

$$\left(\int_{\mathbb{R}} t^2 |f(t)|^2\, dt\right)\left(\int_{\mathbb{R}} w^2 |F(w)|^2\, dw\right) \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f(t)|^2\, dt\right)^2. \quad (4.30)$$

Next, combining (4.29)with the definition of σ_w^2 gives

$$\int_{\mathbb{R}} w^2 |H(w)|^2\, dw = \int_{\mathbb{R}} w^2 |F(w)|^2\, dw. \quad (4.31)$$

Substituting (4.32) into (4.31) yields

$$\sigma_t^2 \sigma_w^2 \geq \frac{1}{4}\left(\int |f(t)|^2\, dt\right)^2,$$

which completes the derivation of the Hartley Heisenberg uncertainty principle.

5. CONCLUTION

This paper has presented the fundamental properties of the Hartley transform, including its inversion formula, Plancherel identity, and core operational rules. The relation between the Hartley and Fourier transforms has been clarified, enabling a unified analytical framework. Two versions of the Heisenberg uncertainty principle for the Hartley transform were established—one derived directly from its kernel structure and the other obtained via its connection to the Fourier transform.

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