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Comparison Of Some Estimation Methods For The Estimators Of Marshall Olkin Distribution With Simulation

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Abstract*.The research comprised multiple simulated tests to determine the relationship between (sample size, distribution parameter value, estimation method, and pollution indivuduales). The experimental findings indicate that the estimator is influenced by sample size, the value of distribution parameter, estimation method, and pollution indivuduales. The results of the mean square error analysis indicate that (robust estimation method) produces the best results with the lowest mean square error, and the best estimation method was (191) of (243) simulation experiments. Additional statistical distributions with additional factors can be performed to demonstrate additional results.*

Keywords: *Marshall Olkin distributions, Probability density function, Cumulative density function, Maximum likelihood estimation method, Robust estimation method, Mean square error.*

Abstrak.Penelitian ini terdiri dari beberapa pengujian simulasi untuk mengetahui hubungan antara (ukuran sampel, nilai parameter distribusi, metode estimasi, dan individu polusi). Temuan eksperimen menunjukkan bahwa penduga dipengaruhi oleh ukuran sampel, nilai parameter distribusi, metode estimasi, dan individu polusi. Hasil analisis mean square error menunjukkan bahwa (metode estimasi Robust) memberikan hasil terbaik dengan mean square error terendah, dan metode estimasi terbaik adalah (191) dari (243) percobaan simulasi. Distribusi statistik tambahan dengan faktor tambahan dapat dilakukan untuk menunjukkan hasil tambahan.

Kata Kunci: Distribusi Marshall Olkin, Fungsi kepadatan probabilitas, Fungsi kepadatan kumulatif, Metode estimasi kemungkinan maksimum, Metode estimasi Robust, Mean square error.

1. INTRODUCTION

The Marshall Olkin family has a broad range of uses, making it critical for a variety of life events. As a result, the presence of methods for estimating the parameters in this family and their functions that are least impacted by contaminants is critical somewhere due to the increasing need for them. The research challenge is that obtaining a sample with a specified distribution is so difficult that failure to do so casts doubt on all of the results of estimating the parameters of the assumed distribution, as well as the associated functions (reliability, survival, failure, entropy). This results in the inability of the system to be reliable in practice, as a result of the presence of vocabulary in various proportions, contaminating the supposed distribution. As a result, much research has been conducted, culminating in the presentation of the estimation method (MLE) by (Jamalizadeh & Kundu, 2013), which comprised the estimation of the four parameters of the distribution and the real data used in the research. (Handique & Chakraborty, 2015) in their research establish a new Marshall family distribution (the Kumarswamy-G family). Additionally, the research included an estimate of the entropy

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function based on real-world data that contained (346 observations). (Jayakumar & Sankaran, 2017)

presented a family of Marshall Olkin distributions based on the binomial distribution and estimated the survival function of the assumed distribution using real data in their research.

Additionally, based on the research of (Muhammad & Liu, 2019), this research discusses the family of Marshall-Olkin truncated distributions and their associated functions (survival function, dependency function, and others). The moment-generating function and its formulas were also presented as part of the research. Another paper by (Williams et al. 2013) includes various simulation experiments to illustrate the influence of good-fit tests and the technique of (weighting Maximum Likelihood Estimators). (Gillariose et al., 2021) presented several families of Marshall distributions in their study, including the (Beta Marshall Olkin family, the Truncated discrete Linnik family, the Weibull Marshall Olkin family, and others), as well as the properties of these families of distributions using their probability density functions.

1. Marshall Olkin distribution

The Marshall-Olkin distribution is a significant statistical distribution that is used to represent continuous variables in a wide variety of real-world situations, physical, biological, and other applications. This distribution is made up of densities (functions of exponential densities) deriving from the same underlying distribution. It uses their partial differentials to compute the quantum function, the characteristic function, the moment generating function, and other properties of these distributions. It is feasible to obtain the requisite family of distributions by relying on continuous primary distributions, such as the Marshall Olkin family for the truncated exponential distribution.

To do this, various fundamental ideas, such as the law of an infinite series sum, are utilized, as indicated by the following formula [1]

$e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \quad x^i$ (1)

Based on a parameter () and the following cumulative distribution function, Marshall Olkin's family has been assumed.

$M(X) = \frac{G(X)}{\propto + \bar{\alpha}\, G(X)} \quad ;\ \alpha \geq 0, \alpha + \bar{\alpha} = 1$ (2)

$G(X)$ represent the cumulative distribution function for any ather distribution

And the probability density function of the Marshall Olkin distribution is given by :

$m(x) = \frac{\propto g(x)}{(\propto + \bar{\alpha} G(X))^2}$ (3)

$g(x)$ represent the probability density function for any ather distribution

And

$$w(x) = \frac{\theta e^{-\theta x}}{1-e^{-\theta}} \qquad (4)$$

$$W(X) = \frac{1-e^{-\theta X}}{1-e^{-\theta}} \quad ; \; 0 < X < 1, \theta \geq 0 \quad (5)$$

the cumulative distribution function for generalized truncated exponential Marshall Olkin distribution will be

$$F(x) = \frac{1-e^{-\theta M(X)}}{1-e^{-\theta}} \quad (6)$$

The probability density function for generalized truncated exponential Marshall Olkin distribution will be

$$f(x) = \frac{\theta m(x) e^{-\theta M(X)}}{(1-e^{-\theta})} \quad (7)$$

And by substituting equation (2) into Equation (6), we get

$$F(x) \frac{1-e^{-\theta \frac{G(X)}{\propto + \bar{\propto} G(X)}}}{1-e^{-\theta}} \qquad (8)$$

And by substituting equations (2 and 3) into Equation (7), we get

$$f(x) = \frac{\theta \alpha g(x) e^{-\theta \frac{G(X)}{\propto + \bar{\propto} G(X)}}}{(1-e^{-\theta})(\propto + \bar{\propto} G(X))^2} \quad (9)$$

And

$$e^{-\theta \frac{G(X)}{\propto + \bar{\propto} G(X)}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \theta^i \, G(X)^i \, (\propto + \bar{\propto} \, G(X))^{-i} \qquad (10)$$

And by substituting equation (10) into Equation (9), we get

$$f(x) = \frac{\theta \alpha g(x) \, \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \theta^i \, G(X)^i \, (\propto + \bar{\propto} \, G(X))^{-i}}{(1 - e^{-\theta})(\propto + \bar{\propto} \, G(X))^2}$$

$$f(x) = \frac{\theta \alpha g(x) \, \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \theta^i \, G(X)^i \, (\propto + \bar{\propto} G(X))^{-(i+2)}}{(1-e^{-\theta})} \quad (11)$$

By applying the following formula on (11)

$$(1 - X)^{-\alpha} = \sum_{i=0}^{\infty} \frac{\Gamma(\propto + i)}{i! \, \Gamma(\alpha)} X^i$$

We will get

$$(\propto + \bar{\propto} \, G(X))^{-(i+2)} = \alpha^{-(i+2)} \sum_{i=0}^{\infty} \frac{(-1)^j}{j!} \frac{(i+j+2)}{(i+2)} \left(\frac{\bar{\alpha}}{\alpha}\right)^j (G(X))^j \quad (12)$$

$$f(x) = \frac{\theta \alpha g(x) \, \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \theta^i \, G(X)^i \, \alpha^{-(i+2)} \, \sum_{i=0}^{\infty} \frac{(-1)^j}{j!} \frac{(i+j+2)}{(i+2)} \left(\frac{\bar{\alpha}}{\alpha}\right)^j (G(X))^j}{(1-e^{-\theta})} \quad (13)$$

There are multiple distributions that can be obtained from the Marshall Olkin exponential distribution family, one of which is The Truncated Exponential Marshall Olkin Exponential Distribution, which is one of the main distributions from the Marshall Olkin exponential family of distributions. The cumulative distribution and the probability density function can be obtained from this distribution by substituting the cumulative distribution and the probability density function with the parameter (λ). As known all the exponential distributions have the cumulative distribution and the probability density function and it takes the following formats.

$G(X) = 1 - e^{-\lambda x}$

$g(x) = \lambda e^{-\lambda x}$

By substituting both the above cumulative function and the probability density function into the formula(8) and (9) respectively we get

Cumulative Function and Probability Density Function of truncated Exponential Marshall-Olkin Exponential Distribution

$$F(X) = \frac{1-e^{-\theta \frac{1-e^{-\lambda x}}{\alpha+\bar{\alpha}\left(1-e^{-\lambda x}\right)}}}{(1-e^{-\theta})} \quad (14)$$

$$f(x) = \frac{\theta\, \alpha\lambda\, e^{-\lambda x} e^{-\theta \frac{1-e^{-\lambda x}}{\alpha+\bar{\alpha}\left(1-e^{-\lambda x}\right)}}}{\left(1-e^{-\theta}\right)\left(\alpha+\bar{\alpha}\left(1-e^{-\lambda x}\right)\right)^2} \quad (15)$$

2. Estimation Method

We will discuss a variety of estimation methodologies for estimating the distribution's parameters, including the Maximum Likelihood and robust estimation Methods.

a.Maximum Likelihood Estimators(MLE)

Maximum Likelihood Estimators are a widely used estimating technique. They are predicated on the availability of a random sample of size (n).

$$l = \prod_{i=1}^{n} f(x_i)$$

$$l = \prod_{i=1}^{n} \frac{\theta\, \alpha\lambda\, e^{-\lambda x} e^{-\theta \frac{1-e^{-\lambda x}}{\alpha+\bar{\alpha}\left(1-e^{-\lambda x}\right)}}}{\left(1-e^{-\theta}\right)\left(\alpha+\bar{\alpha}\left(1-e^{-\lambda x}\right)\right)^2}$$

$$l = \left(\frac{\theta\alpha\lambda}{1-e^{-\theta}}\right)^{n} e^{-\lambda \sum_{i=1}^{n} x_i}\, e^{-\theta \sum_{i=1}^{n} \frac{1-e^{-\lambda x_i}}{\alpha+\bar{\alpha}\left(1-e^{-\lambda x_i}\right)}} \prod_{i=1}^{n} \left(\alpha+\bar{\alpha}\left(1-e^{-\lambda x}\right)\right)^{-2}$$

Tacking the Logarithm function we get

$$L(\alpha,\theta,\lambda) = n ln\left(\frac{\theta\alpha\lambda}{1-e^{-\theta}}\right) - \lambda \sum_{i=1}^{n} x_i - \theta \sum_{i=1}^{n} \frac{1-e^{-\lambda x_i}}{\alpha+\bar{\alpha}\left(1-e^{-\lambda x_i}\right)} - 2\sum_{i=1}^{n} \ln\left(\alpha+\bar{\alpha}\left(1-e^{-\lambda x_i}\right)\right) \quad (16)$$

By taking the partial derivatives of the previous formula concerning $(\alpha.\theta,\lambda)$ to be

$$\frac{\partial L(\propto,\theta,\lambda)}{\partial\alpha}=\frac{n}{\alpha}-\theta\sum_{i=1}^{n}\frac{(e^{-\lambda x_i}-1)e^{-\lambda x_i}}{(\alpha+(1-\alpha)\,(1-e^{-\lambda x_i}))^2}-2\sum_{i=1}^{n}\frac{e^{-\lambda x_i}}{\alpha+(1-\alpha)\,(1-e^{-\lambda x_i})} \quad (17)$$

$$\frac{\partial L(\propto,\theta,\lambda)}{\partial\theta}=\frac{n}{\theta}-\frac{ne^{-\theta}}{(1-e^{-\theta})}-\sum_{i=1}^{n}\frac{1-e^{-\lambda x_i}}{\alpha+\bar{\alpha}\,(1-e^{-\lambda x_i})} \quad (18)$$

$$\frac{\partial L(\propto,\theta,\lambda)}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i-\theta\sum_{i=1}^{n}\frac{x_ie^{-\lambda x_i}}{\alpha+\bar{\alpha}\,(1-e^{-\lambda x_i})}-\frac{\bar{\alpha}\,(x_ie^{-\lambda x_i})(1-e^{-\lambda x_i})}{\left(\alpha+\bar{\alpha}\,(1-e^{-\lambda x_i})\right)^2}-2\sum_{i=1}^{n}\frac{\bar{\alpha}xi\,e^{-\lambda x_i}}{\alpha+\bar{\alpha}\,(\;\;1-e^{-\lambda x_i})}$$

(19)

And by making the all partial derivative equal to zero we get

$$0=\frac{n}{\alpha}-\theta\sum_{i=1}^{n}\frac{(e^{-\lambda x_i}-1)e^{-\lambda x_i}}{(\alpha+(1-\alpha)\,(1-e^{-\lambda x_i}))^2}-2\sum_{i=1}^{n}\frac{e^{-\lambda x_i}}{\alpha+(1-\alpha)\,(1-e^{-\lambda x_i})} \quad (20)$$

$$0=\frac{n}{\theta}-\frac{ne^{-\theta}}{(1-e^{-\theta})}-\sum_{i=1}^{n}\frac{1-e^{-\lambda x_i}}{\alpha+\bar{\alpha}\,(1-e^{-\lambda x_i})} \quad (21)$$

$$0=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i-\theta\sum_{i=1}^{n}\frac{x_ie^{-\lambda x_i}}{\alpha+\bar{\alpha}\,(1-e^{-\lambda x_i})}-\frac{\bar{\alpha}\,(x_ie^{-\lambda x_i})(1-e^{-\lambda x_i})}{\left(\alpha+\bar{\alpha}\,(1-e^{-\lambda x_i})\right)^2}-2\sum_{i=1}^{n}\frac{\bar{\alpha}x_i\,e^{-\lambda xi}}{\alpha+\bar{\alpha}\,(\;\;1-e^{-\lambda x_i})}$$

(22)

Prior formulations and their equality to zero cannot be solved using conventional methods for obtaining estimators of the distribution's parameters, and hence the estimators must be determined using iterative numerical methods.

3. Robust Estimation Method(REM)

robust estimation method depend on sensing high deviations in some individuals of the sample and finding the estimator that is based on reducing the effect of these deviations

By equating the cumulative distribution function of the Marshall-Olkin exponential general truncated exponential distribution to the ratio, we get

$$m_i\,(\alpha,\theta,\lambda)=-\frac{1}{\lambda}\;ln\left[1+\frac{\alpha}{\frac{\theta}{ln\left[1-pi\left(1-e^{-\theta}\right)\right]}+(1-\alpha)}\right]$$

(23) with

$x_i = m_i\;\;(\alpha,\theta,\lambda)+e$

Such that (x_i) as a function with respect to (α,θ,λ) And (ei) represent the error term and caclated by tacking difference between a real function and an estimated function with percentile And ei=$\frac{ui}{s}$ with S=$\frac{MAD(u)}{0.6745}$ And (ui) the error term $MAD(u)=MAD=(u_1,u_2,\ldots u_3)=Median[|u-Median(u)|]$By tacking Tukey's Bisquare and Huber's weight we get

$$\rho(e_i)=\begin{cases}1-\left(1-\left(\frac{e_i}{a}\right)^2\right)^3 & |e_i|\le a\\ 1 & |e_i|\;>a\end{cases} \quad (24)$$

And

$$\acute{\rho}(e_i)=\begin{cases}\frac{6e_i}{a^2}\left(1-\left(\frac{e_i}{a}\right)^2\right)^2 & |e_i|\le a\\ 0 & |e_i|\;>a\end{cases} \quad (25)$$

When (α=4.685) And (Huber's weight) will be $\rho(e_i) = \begin{cases} \frac{1}{2}{e_i}^2 & |e_i| \le a \\ a|e_i| - \frac{1}{2}\,a^2 & |e_i| > a \end{cases}$ (26) and

$\acute{\rho}(e_i) = \begin{cases} e_i & |e_i| \le a \\ a\ sign\ e_i & |e_i| > a \end{cases}$ (27)$\alpha = 1.345$)

When The robust estimators according to this method will be obtained by tacking (summation for product of the derivative of the function (ρ) in the partial derivatives of the function mi(α, θ, λ) for the parameters and λ θ, α respectively

$$\sum_{i=1}^{n} \acute{\rho}(e_i)\,\frac{\partial m_i(\propto,\theta,\lambda)}{\partial \alpha} = \sum_{i=1}^{n} \acute{\rho}(e_i)\,(-\frac{1}{\lambda})\left[\frac{1}{1+\frac{\alpha}{\frac{\theta}{ln\left[1-p_i\left(1-e^{-\theta}\right)\right]}+(1-\alpha)}}\right]\frac{\frac{\theta}{ln\left[1-p_i(1-e^{-\theta})\right]}}{\left(\frac{\theta}{ln\left[1-p_i\left(1-e^{-\theta}\right)\right]}+(1-\alpha)\right)^2} + 1 \quad (28)$$

$$\sum_{i=1}^{n} \acute{\rho}(e_i)\,\frac{\partial m_i(\propto,\theta,\lambda)}{\partial \theta} = \sum_{i=1}^{n} \acute{\rho}(e_i)\,(-\frac{1}{\lambda})\left[\frac{1}{1+\frac{\alpha}{\frac{\theta}{ln\left[1-p_i\left(1-e^{-\theta}\right)\right]}+(1-\alpha)}}\right]\frac{\frac{(-\alpha)ln\left[1-p_i\left(1-e^{-\theta}\right)\right]-\theta\frac{1}{1-p_i\left(1-e^{-\theta}\right)}(-p_i e^{-\theta})}{\left[\ln[1-p_i\left(1-e^{-\theta}\right)]\right]^2}}{\left[\frac{\theta}{ln\left[1-p_i(1-e^{-\theta})\right]}+(1-\alpha)\right]^2}$$

(29)

$$\sum_{i=1}^{n} \acute{\rho}(e_i)\,\frac{\partial m_i(\propto,\theta,\lambda)}{\partial \lambda} = \ln\left[1+\frac{\alpha}{\frac{\theta}{ln\left[1-p_i\left(1-e^{-\theta}\right)\right]}+(1-\alpha)}\right](\frac{1}{\lambda^2}) \quad (3$$

And by setting the previous formulas equal to zero

1 (31)

$$0= \sum_{i=1}^{n} \acute{\rho}(e_i)\,(-\frac{1}{\lambda})\left[\frac{1}{1+\frac{\alpha}{\frac{\theta}{ln\left[1-p_i\left(1-e^{-\theta}\right)\right]}+(1-\alpha)}}\right]\frac{\frac{\theta}{ln\left[1-p_i(1-e^{-\theta})\right]}}{\left(\frac{\theta}{ln\left[1-p_i\left(1-e^{-\theta}\right)\right]}+(1-\alpha)\right)^2} +$$

$$0=\sum_{i=1}^{n} \acute{\rho}(e_i)\,(-\frac{1}{\lambda})\left[\frac{1}{1+\frac{\alpha}{\frac{\theta}{ln\left[1-p_i\left(1-e^{-\theta}\right)\right]}+(1-\alpha)}}\right]\frac{\frac{(-\alpha)ln\left[1-p_i\left(1-e^{-\theta}\right)\right]-\theta\frac{1}{1-p_i\left(1-e^{-\theta}\right)}(-p_i e^{-\theta})}{\left[\ln[1-p_i\left(1-e^{-\theta}\right)]\right]^2}}{\left[\frac{\theta}{ln\left[1-p_i(1-e^{-\theta})\right]}+(1-\alpha)\right]^2}$$

(32)

$$0= \ln\left[1+\frac{\alpha}{\frac{\theta}{ln\left[1-p_i\left(1-e^{-\theta}\right)\right]}+(1-\alpha)}\right](\frac{1}{\lambda^2})$$

(33)

The previous formulas are non-linear functions, so the estimated parameters will be obtained by using numerical methods and solving their simultaneous equations

4. Polluted Distribution

A polluted value appears to be (more or less) distinct from the other values and can be determined by its reading is significantly higher or lower than the other values within the same sample [10]. The next illustration depicts the polluted values as red circles, while the remaining values are shown in green.

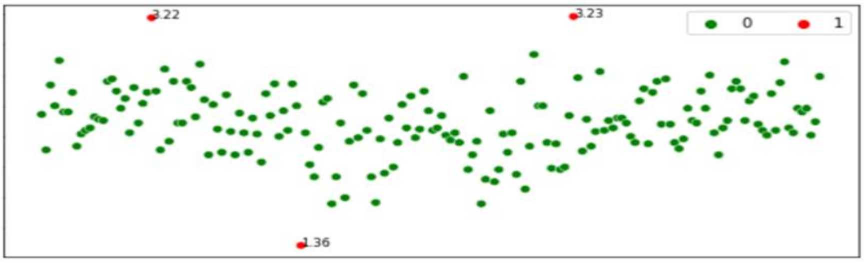

Figure (1) The polluted data [10]

Statistical distributions are mathematically represented by a specific function that is used to characterize their behavior, particularly when the movement of the random variable within the assumed distribution is known. (Determination of the probability density function and the cumulative distribution function, as well as their effect on parameter estimation. Outliers are the primary contributors to the pollution of a given statistical distribution. Outliers are data that are significantly different from the majority, implying a measurement error. By increasing the number of pollutant values, the distribution can become a mixture of many distributions, and in big samples, the effect of pollutants diminishes, particularly when they differ from other sample observations or have small numbers compared to the sample size.

We generate the outliers from the uniform distribution with parameters $(\bar{x} + 4s; \bar{x} + 7s)$; where $\bar{x} and\ s$ is the mean and standard deviation of the generated sample .

We take one outlier for the small sample size (10), tow outliers for the moderate sample size (50) and five outliers for the large sample (100) (Orabi et al., 2021).

5. Compare Criteria

Numerous criteria can be used to assess the various estimating methods to identify which method is the best or worse by comparing the real and estimated values of the distribution parameters to be estimated. Certain standards include special formulas that allow for the determination of the optimal estimation method after a certain number of iterations of simulation experiments based on generating the specified statistical distribution under specified conditions. Among the comparison criteria that may be used to compare various estimating techniques is the mean square error (MSE), which has the following general formula [1]:

$$MSE = \frac{\sum_{i=1}^{itr}[\hat{\theta}_i - \theta]^2}{itr} \quad \dots (33)$$

6. Simulation Experiments

To execute simulation experiments, a program python was adopted according to the following settings: Sample Size ($n = 10,50,100$), the value of the first parameter to be ($\alpha = 0.25,0.50,0.75$), the value of the second parameter to be ($\theta = 1,1.5,2$), the value of the Third parameter ($\lambda = 1,2,3$),and the outlier(polluted indivuduales) will be(1,2,5). Simulation experiments are adopted to ensure obtaining duplicate samples according to the conditions of the assumed distribution and according to the initial information it contains, based on the random generation function (Randomize), which as uniform distributed within the period [0, 1], which is included in the calculations of individual generation each simulation experiment was repeated 1000 times..

7. Experimental results

The entire number of simulation experiments was indicated by the factors used, with a total of (243), and the tables contain two major sections: estimator values and mean squared error. The entire number of simulation experiments was represented by the factors used, where the total number of simulation experiments was (243), and the tables contain two major components: estimator values and mean squared error.

Table (1) shows the estimators' simulation results and the mean square error of The first parameter (α)

α	θ	λ	n	MLE	REM	MSE_{MLE}	MSE_{REM}	$Best$
0.25	1	1	10	0.826933665	0.28656321	0.332852453	0.001336868	2
0.25	1	1	50	0.370731292	0.251735026	0.014576045	3.01032E-06	2
0.25	1	1	100	0.319502144	0.250445817	0.004830548	1.98753E-07	2
0.25	1	2	10	0.262458647	0.309276443	0.000155218	0.003513697	1
0.25	1	2	50	0.296314528	0.254175989	0.002145035	1.74389E-05	2
0.25	1	2	100	0.249715104	0.250294026	8.11659E-08	8.64512E-08	1
0.25	1	3	10	0.223173754	0.323196039	0.000719647	0.00535766	1
0.25	1	3	50	0.351756038	0.257451913	0.010354291	5.5531E-05	2
0.25	1	3	100	0.323320047	0.250407417	0.005375829	1.65989E-07	2
0.25	1.5	1	10	0.315909995	0.34121719	0.004344127	0.008320576	1
0.25	1.5	1	50	0.26859236	0.250983574	0.000345676	9.67418E-07	2
0.25	1.5	1	100	0.247378892	0.250087917	6.87021E-06	7.72935E-09	2
0.25	1.5	2	10	0.361180677	0.330962285	0.012361143	0.006554892	2
0.25	1.5	2	50	0.317220331	0.254552466	0.004518573	2.07249E-05	2
0.25	1.5	2	100	0.261143403	0.25063539	0.000124175	4.03721E-07	2
0.25	1.5	3	10	0.310102465	0.318070658	0.003612306	0.004633614	1
0.25	1.5	3	50	0.359105316	0.25496752	0.01190397	2.46763E-05	2
0.25	1.5	3	100	0.262298274	0.250716555	0.000151248	5.13451E-07	2
0.25	2	1	10	0.490204777	0.332869157	0.057698335	0.006867297	2
0.25	2	1	50	0.290503401	0.250953862	0.001640525	9.09853E-07	2
0.25	2	1	100	0.24881147	0.250343527	1.4126E-06	1.18011E-07	2
0.25	2	2	10	0.286664418	0.293965869	0.00134428	0.001932998	1
0.25	2	2	50	0.241532368	0.253928883	7.17008E-05	1.54361E-05	2
0.25	2	2	100	0.274500661	0.250818155	0.000600282	6.69377E-07	2
0.25	2	3	10	0.278703909	0.300054651	0.000823914	0.002505468	1

0.25	2	3	50	0.246478846	0.256359164	1.23985E-05	4.0439E-05	1
0.25	2	3	100	0.259897655	0.250197522	9.79636E-05	3.9015E-08	2
0.5	1	1	10	0.615614423	0.591704041	0.013366695	0.008409631	2
0.5	1	1	50	0.508484508	0.502374803	7.19869E-05	5.63969E-06	2
0.5	1	1	100	0.497764799	0.50031717	4.99612E-06	1.00597E-07	2
0.5	1	2	10	0.564754405	0.505037695	0.004193133	2.53784E-05	2
0.5	1	2	50	0.57605208	0.509895722	0.005783919	9.79253E-05	2
0.5	1	2	100	0.535386131	0.500941608	0.001252178	8.86626E-07	2
0.5	1	3	10	0.902781121	0.563790664	0.162232631	0.004069249	2
0.5	1	3	50	0.566309427	0.504079849	0.00439694	1.66452E-05	2
0.5	1	3	100	0.499739591	0.500056266	6.78128E-08	3.16588E-09	2
0.5	1.5	1	10	0.999069623	0.586538985	0.249070489	0.007488996	2
0.5	1.5	1	50	0.534221359	0.509547546	0.001171101	9.11556E-05	2
0.5	1.5	1	100	0.532635077	0.500178843	0.001065048	3.1985E-08	2
0.5	1.5	2	10	0.597444783	0.522415977	0.009495486	0.000502476	2
0.5	1.5	2	50	0.50082098	0.505445997	6.74007E-07	2.96589E-05	1
0.5	1.5	2	100	0.620482377	0.500523055	0.014516003	2.73586E-07	2
0.5	1.5	3	10	0.795067648	0.535789303	0.087064917	0.001280874	2
0.5	1.5	3	50	0.50822442	0.503619382	6.76411E-05	1.30999E-05	2
0.5	1.5	3	100	0.602694303	0.500138307	0.01054612	1.9129E-08	2
0.5	2	1	10	0.920819547	0.599855378	0.177089091	0.009971097	2
0.5	2	1	50	0.503623994	0.501964581	1.31333E-05	3.85958E-06	2
0.5	2	1	100	0.493302011	0.500022069	4.48631E-05	4.87056E-10	2
0.5	2	2	10	0.619929626	0.51436091	0.014383115	0.000206236	2
0.5	2	2	50	0.611022588	0.505533473	0.012326015	3.06193E-05	2
0.5	2	2	100	0.527778449	0.500472863	0.000771642	2.236E-07	2
0.5	2	3	10	0.675855013	0.588538989	0.030924985	0.007839153	2
0.5	2	3	50	0.607034159	0.505098917	0.011456311	2.5999E-05	2
0.5	2	3	100	0.534013851	0.500156292	0.001156942	2.44273E-08	2
0.75	1	1	10	0.744795652	0.80942269	2.70852E-05	0.003531056	1
0.75	1	1	50	1.112362069	0.752967431	0.131306269	8.80565E-06	2
0.75	1	1	100	0.749531671	0.750464157	2.19332E-07	2.15442E-07	2
0.75	1	2	10	1.791035319	0.780005794	1.083754536	0.000900348	2
0.75	1	2	50	0.75294887	0.752532875	8.69583E-06	6.41546E-06	2
0.75	1	2	100	0.887737441	0.750454898	0.018971603	2.06932E-07	2
0.75	1	3	10	3.296482232	0.848506078	6.484571756	0.009703447	2
0.75	1	3	50	0.753655383	0.756218955	1.33618E-05	3.86754E-05	1
0.75	1	3	100	0.901834561	0.750079337	0.023053734	6.29436E-09	2
0.75	1.5	1	10	0.779512582	0.835907415	0.000870993	0.007380084	1
0.75	1.5	1	50	0.739456367	0.751795141	0.000111168	3.22253E-06	2
0.75	1.5	1	100	0.902066858	0.75072056	0.023124329	5.19207E-07	2
0.75	1.5	2	10	1.007364288	0.753302962	0.066236377	1.09096E-05	2
0.75	1.5	2	50	1.033985497	0.758041188	0.080647762	6.46607E-05	2
0.75	1.5	2	100	0.745504921	0.750303587	2.02057E-05	9.2165E-08	2
0.75	1.5	3	10	0.896803136	0.839825399	0.021551161	0.008068602	2
0.75	1.5	3	50	1.029812042	0.758573429	0.078294779	7.35037E-05	2
0.75	1.5	3	100	0.82574479	0.750642808	0.005737273	4.13203E-07	2
0.75	2	1	10	1.354152216	0.831035331	0.3649999	0.006566725	2
0.75	2	1	50	0.792940743	0.75680144	0.001843907	4.62596E-05	2
0.75	2	1	100	0.845309435	0.750304067	0.009083888	9.24565E-08	2
0.75	2	2	10	0.752962631	0.832064301	8.77718E-06	0.006734549	1
0.75	2	2	50	0.88435794	0.754037325	0.018052056	1.63E-05	2
0.75	2	2	100	0.737156423	0.750998268	0.000164957	9.9654E-07	2
0.75	2	3	10	1.052547631	0.799392708	0.091535069	0.00243964	2
0.75	2	3	50	0.739612994	0.754979722	0.00010789	2.47976E-05	2
0.75	2	3	100	0.867973217	0.750120846	0.01391768	1.46037E-08	2

According to the table (1), the estimator of the first parameter was affected by both the real values of the distribution parameters and the pollution and the sample size.

Table (2) shows the estimators' simulation results and the mean square error of The second parameter (θ)

α	θ	λ	n	MLE	REM	MSE_{MLE}	MSE_{REM}	$Best$
0.25	1	1	10	0.866251392	1.039191533	0.01788869	0.001535976	2
0.25	1	1	50	0.944809892	1.005734063	0.003045948	3.28795E-05	2
0.25	1	1	100	0.864116797	1.00004278	0.018464245	1.83009E-09	2
0.25	1	2	10	0.910612243	1.059870195	0.007990171	0.00358444	2
0.25	1	2	50	0.766634398	1.007878948	0.054459504	6.20778E-05	2
0.25	1	2	100	0.959415335	1.000462508	0.001647115	2.13914E-07	2
0.25	1	3	10	0.924075186	1.027242279	0.005764577	0.000742142	2
0.25	1	3	50	0.626947081	1.002518951	0.139168481	6.34511E-06	2
0.25	1	3	100	0.985960428	1.000383565	0.00019711	1.47122E-07	2
0.25	1.5	1	10	1.131508325	1.582309678	0.135786115	0.006774883	2
0.25	1.5	1	50	1.641641434	1.508266741	0.020062296	6.8339E-05	2
0.25	1.5	1	100	1.564351042	1.500822046	0.004141057	6.7576E-07	2
0.25	1.5	2	10	0.536654539	1.592517756	0.928034477	0.008559535	2
0.25	1.5	2	50	1.471792896	1.502027837	0.000795641	4.11212E-06	2
0.25	1.5	2	100	1.342595084	1.500940857	0.024776308	8.85211E-07	2
0.25	1.5	3	10	0.502192218	1.510635996	0.995620369	0.000113124	2
0.25	1.5	3	50	1.42173765	1.5032192	0.006124995	1.03633E-05	2
0.25	1.5	3	100	1.671328982	1.500958592	0.02935362	9.18899E-07	2
0.25	2	1	10	1.845146007	2.075371362	0.023979759	0.005680842	2
0.25	2	1	50	1.840020326	2.002696696	0.025593496	7.27217E-06	2
0.25	2	1	100	2.108373251	2.000745158	0.011744761	5.5526E-07	2
0.25	2	2	10	0.847574369	2.060587071	1.328084835	0.003670793	2
0.25	2	2	50	1.601455609	2.000114409	0.158837631	1.30894E-08	2
0.25	2	2	100	1.687395882	2.000988754	0.097721335	9.77635E-07	2
0.25	2	3	10	0.833050485	2.032701589	1.361771171	0.001069394	2
0.25	2	3	50	1.667722308	2.006353837	0.110408465	4.03713E-05	2
0.25	2	3	100	1.827731227	2.000221258	0.02967653	4.8955E-08	2
0.5	1	1	10	0.757055658	1.037196778	0.059021953	0.0013836	2
0.5	1	1	50	0.822898732	1.005743443	0.031364859	3.29871E-05	2
0.5	1	1	100	0.831987919	1.000029796	0.02822806	8.87784E-10	2
0.5	1	2	10	1.458875459	1.006326675	0.210566687	4.00268E-05	2
0.5	1	2	50	1.025294047	1.00452346	0.000639789	2.04617E-05	2
0.5	1	2	100	0.814102	1.00073021	0.034558066	5.33206E-07	2
0.5	1	3	10	1.049535861	1.09857364	0.002453802	0.009716763	1
0.5	1	3	50	1.104678	1.006438853	0.010957484	4.14588E-05	2
0.5	1	3	100	0.863967112	1.000694342	0.018504947	4.8211E-07	2
0.5	1.5	1	10	0.422885905	1.530084515	1.160174773	0.000905078	2
0.5	1.5	1	50	1.587532967	1.508878537	0.00766202	7.88284E-05	2
0.5	1.5	1	100	1.290879611	1.500494091	0.043731337	2.44126E-07	2
0.5	1.5	2	10	1.685818728	1.565959409	0.0345286	0.004350644	2
0.5	1.5	2	50	0.719572216	1.508619705	0.609067525	7.42993E-05	2
0.5	1.5	2	100	1.447035884	1.500832063	0.002805198	6.92328E-07	2
0.5	1.5	3	10	1.27469842	1.525595283	0.050760802	0.000655119	2
0.5	1.5	3	50	1.448678074	1.503567325	0.00263394	1.27258E-05	2
0.5	1.5	3	100	1.466041475	1.500653784	0.001153181	4.27434E-07	2
0.5	2	1	10	0.988251427	2.092527911	1.023635175	0.008561414	2
0.5	2	1	50	2.26860635	2.008937002	0.072149371	7.987E-05	2
0.5	2	1	100	2.217082089	2.000873104	0.047124633	7.62311E-07	2
0.5	2	2	10	1.261879466	2.058142185	0.544821923	0.003380514	2
0.5	2	2	50	1.333116946	2.002698098	0.444733007	7.27973E-06	2

0.5	2	2	100	1.666206404	2.000281953	0.111418165	7.94977E-08	2
0.5	2	3	10	3.344711696	2.024426317	1.808249547	0.000596645	2
0.5	2	3	50	1.694022382	2.00643829	0.093622303	4.14516E-05	2
0.5	2	3	100	1.886584442	2.000711646	0.012863089	5.0644E-07	2
0.75	1	1	10	1.921180053	1.052156068	0.848572691	0.002720255	2
0.75	1	1	50	0.764589489	1.005113021	0.055418109	2.6143E-05	2
0.75	1	1	100	1.052552715	1.000439908	0.002761788	1.93519E-07	2
0.75	1	2	10	0.659334038	1.047975112	0.116053298	0.002301611	2
0.75	1	2	50	0.882832259	1.008853213	0.01372828	7.83794E-05	2
0.75	1	2	100	0.970181646	1.000635164	0.000889134	4.03434E-07	2
0.75	1	3	10	0.898164997	1.092230126	0.010370368	0.008506396	2
0.75	1	3	50	0.925661409	1.002262746	0.005526226	5.12002E-06	2
0.75	1	3	100	0.928550681	1.000335579	0.005105005	1.12613E-07	2
0.75	1.5	1	10	1.607384842	1.509469462	0.011531504	8.96707E-05	2
0.75	1.5	1	50	1.362660528	1.502970711	0.018862131	8.82512E-06	2
0.75	1.5	1	100	1.6007413	1.500173139	0.01014881	2.9977E-08	2
0.75	1.5	2	10	0.588687056	1.575788692	0.830491281	0.005743926	2
0.75	1.5	2	50	1.039259018	1.502606593	0.212282252	6.79433E-06	2
0.75	1.5	2	100	1.3885948	1.50009154	0.012411118	8.37956E-09	2
0.75	1.5	3	10	0.562633329	1.59566474	0.878656276	0.009151742	2
0.75	1.5	3	50	1.314288559	1.503446394	0.034488739	1.18776E-05	2
0.75	1.5	3	100	1.512147579	1.500585106	0.000147564	3.42349E-07	2
0.75	2	1	10	1.713817544	2.001478291	0.081900398	2.18534E-06	2
0.75	2	1	50	2.028814336	2.009167634	0.000830266	8.40455E-05	2
0.75	2	1	100	1.887039649	2.000128187	0.012760041	1.64318E-08	2
0.75	2	2	10	0.891085966	2.007522249	1.229690335	5.65842E-05	2
0.75	2	2	50	1.737397591	2.006927754	0.068960025	4.79938E-05	2
0.75	2	2	100	1.928561094	2.000056811	0.005103517	3.22753E-09	2
0.75	2	3	10	1.552265113	2.055292893	0.200466529	0.003057304	2
0.75	2	3	50	1.668380852	2.006537086	0.109971259	4.27335E-05	2
0.75	2	3	100	1.598565276	2.00085656	0.161149837	7.33696E-07	2

According to the table (2), the estimator of the second parameter was affected by both the real values of the distribution parameters and the pollution and the sample size.

Table (3) shows the estimators' simulation results and the mean square error of The third parameter (λ)

α	θ	λ	n	MLE	REM	MSE_{MLE}	MSE_{REM}	$Best$
0.25	1	1	10	1.077577	1.036281	0.006018209	0.00131631	2
0.25	1	1	50	1.007817	1.00628	6.11115E-05	3.94422E-05	2
0.25	1	1	100	1.001734	1.001097	3.00765E-06	1.20333E-06	2
0.25	1	2	10	2.038841	2.05779	0.001508661	0.003339661	1
0.25	1	2	50	2.002723	2.006442	7.41461E-06	4.15033E-05	1
0.25	1	2	100	2.000508	2.001792	2.57651E-07	3.21281E-06	1
0.25	1	3	10	3.02465	3.078281	0.000607604	0.006127903	1
0.25	1	3	50	3.003078	3.004119	9.47252E-06	1.69655E-05	1
0.25	1	3	100	3.001144	3.001699	1.30967E-06	2.88569E-06	1
0.25	1.5	1	10	1.066279	1.035686	0.004392897	0.001273509	2
0.25	1.5	1	50	1.009221	1.002847	8.50239E-05	8.1072E-06	2
0.25	1.5	1	100	1.001153	1.001188	1.32912E-06	1.41116E-06	1
0.25	1.5	2	10	2.052289	2.016732	0.002734091	0.000279959	2
0.25	1.5	2	50	2.004691	2.003161	2.20022E-05	9.99012E-06	2
0.25	1.5	2	100	2.000835	2.000908	6.97526E-07	8.23846E-07	1
0.25	1.5	3	10	3.047287	3.074047	0.002236059	0.005482971	1
0.25	1.5	3	50	3.004142	3.007277	1.71565E-05	5.29549E-05	1

0.25	1.5	3	100	3.00058	3.000234	3.35931E-07	5.47039E-08	2
0.25	2	1	10	1.01129	1.002448	0.000127471	5.99486E-06	2
0.25	2	1	50	1.005261	1.001535	2.76788E-05	2.35604E-06	2
0.25	2	1	100	1.001239	1.00051	1.53527E-06	2.59712E-07	2
0.25	2	2	10	2.039031	2.091284	0.001523421	0.008332732	1
0.25	2	2	50	2.005413	2.005254	2.92986E-05	2.75998E-05	2
0.25	2	2	100	2.000253	2.000043	6.42313E-08	1.86225E-09	2
0.25	2	3	10	3.081225	3.053798	0.006597526	0.002894195	2
0.25	2	3	50	3.006286	3.000944	3.95096E-05	8.91846E-07	2
0.25	2	3	100	3.001022	3.001425	1.0442E-06	2.02935E-06	1
0.5	1	1	10	1.09964	1.001982	0.009928061	3.92822E-06	2
0.5	1	1	50	1.003476	1.005466	1.20852E-05	2.9877E-05	1
0.5	1	1	100	1.000896	1.000568	8.01921E-07	3.23067E-07	2
0.5	1	2	10	2.044421	2.0172	0.001973264	0.000295832	2
0.5	1	2	50	2.008551	2.000098	7.31135E-05	9.55538E-09	2
0.5	1	2	100	2.000362	2.001924	1.30986E-07	3.70009E-06	1
0.5	1	3	10	3.003158	3.084941	9.97054E-06	0.007214999	1
0.5	1	3	50	3.00503	3.005948	2.53049E-05	3.53795E-05	1
0.5	1	3	100	3.000722	3.00129	5.21529E-07	1.66431E-06	1
0.5	1.5	1	10	1.012986	1.041141	0.000168638	0.001692585	1
0.5	1.5	1	50	1.006235	1.007101	3.88753E-05	5.04217E-05	1
0.5	1.5	1	100	1.001776	1.000487	3.15368E-06	2.36908E-07	2
0.5	1.5	2	10	2.070951	2.052999	0.00503411	0.002808871	2
0.5	1.5	2	50	2.00698	2.000568	4.87196E-05	3.22254E-07	2
0.5	1.5	2	100	2.00091	2.001235	8.2818E-07	1.52583E-06	1
0.5	1.5	3	10	3.031163	3.024824	0.000971131	0.000616214	2
0.5	1.5	3	50	3.003497	3.002321	1.22299E-05	5.38928E-06	2
0.5	1.5	3	100	3.000728	3.0013	5.30174E-07	1.69025E-06	1
0.5	2	1	10	1.006169	1.096633	3.8062E-05	0.009337952	1
0.5	2	1	50	1.003456	1.009997	1.19464E-05	9.99331E-05	1
0.5	2	1	100	1.000681	1.00176	4.63657E-07	3.0993E-06	1
0.5	2	2	10	2.094359	2.022398	0.008903696	0.000501659	2
0.5	2	2	50	2.002229	2.00031	4.96833E-06	9.60654E-08	2
0.5	2	2	100	2.000363	2.00174	1.32023E-07	3.02753E-06	1
0.5	2	3	10	3.054299	3.00709	0.00294842	5.02728E-05	2
0.5	2	3	50	3.005042	3.00678	2.54225E-05	4.59672E-05	1
0.5	2	3	100	3.000695	3.001282	4.83674E-07	1.64346E-06	1
0.75	1	1	10	1.053493	1.052758	0.002861452	0.002783455	2
0.75	1	1	50	1.002831	1.008915	8.01716E-06	7.94763E-05	1
0.75	1	1	100	1.001348	1.000001	1.81587E-06	1.71951E-12	2
0.75	1	2	10	2.057752	2.092596	0.003335254	0.008574073	1
0.75	1	2	50	2.009113	2.005385	8.30525E-05	2.89948E-05	2
0.75	1	2	100	2.00168	2.000825	2.82125E-06	6.80507E-07	2
0.75	1	3	10	3.022455	3.060564	0.000504237	0.003667945	1
0.75	1	3	50	3.00865	3.006753	7.48277E-05	4.55994E-05	2
0.75	1	3	100	3.000102	3.00018	1.03642E-08	3.24022E-08	1
0.75	1.5	1	10	1.081191	1.020516	0.006592027	0.000420888	2
0.75	1.5	1	50	1.000448	1.007319	2.00907E-07	5.35726E-05	1
0.75	1.5	1	100	1.000976	1.001587	9.52743E-07	2.51943E-06	1
0.75	1.5	2	10	2.098065	2.01604	0.009616678	0.000257292	2
0.75	1.5	2	50	2.003593	2.00829	1.29094E-05	6.87171E-05	1
0.75	1.5	2	100	2.001296	2.001072	1.67849E-06	1.14903E-06	2
0.75	1.5	3	10	3.095445	3.068082	0.009109778	0.004635172	2
0.75	1.5	3	50	3.007764	3.003461	6.02776E-05	1.19761E-05	2
0.75	1.5	3	100	3.000407	3.000043	1.65827E-07	1.86225E-09	2
0.75	2	1	10	1.05331	1.062311	0.002841947	0.003882697	1
0.75	2	1	50	1.009278	1.000081	8.6089E-05	6.59044E-09	2
0.75	2	1	100	1.000993	1.000624	9.85602E-07	3.88857E-07	2

0.75	2	2	10	2.04784	2.066787	0.002288654	0.004460472	1
0.75	2	2	50	2.004438	2.003397	1.96952E-05	1.15428E-05	2
0.75	2	2	100	2.001873	2.001984	3.50998E-06	3.93768E-06	1
0.75	2	3	10	3.07051	3.024207	0.004971681	0.000585996	2
0.75	2	3	50	3.007139	3.00226	5.09649E-05	5.10746E-06	2
0.75	2	3	100	3.00077	3.001181	5.93042E-07	1.39506E-06	1

According to the table (3), the estimator of the third parameter was affected by both the real values of the distribution parameters and the pollution and the sample size.

By and large, the most accurate estimating method was (robust estimation method) combined with (81) simulation experiments. Table (1) indicates that the minimum mean square error for MLE is (6.78128E-08) and for REM is (4.87056E-10) and the best estimator for first paremeter was for REM method .

By and large, the most accurate estimating method was (robust estimation method) combined with (81) simulation experiments. Table (2) indicates that the minimum mean square error for MLE is (0.000147564) and for REM is (8.87784E-10) and the best estimator for second paremeter was for REM method .

By and large, the most accurate estimating method was (robust estimation method) combined with (81) simulation experiments. Table (3) indicates that the minimum mean square error for MLE is (1.03642E-08) and for REM is (1.71951E-12) and the best estimator for second paremeter was for REM method .

2. CONCLUSIONS

Several conclusions and proposals emerged as a result of simulation trials with varying distribution parameters, sample size, and pollution variabels . The ability of the estimation methods to display the estimated values of the Marshall-Olkin distribution parameters is generally closer to the true values than the mean square error values are. The optimal strategy is (REM), as it achieved the best outcomes in the majority of simulation studies. The estimation findings were influenced by both the sample size and the true values of the distribution parameters and the pollution variabels.

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