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Kronecker Method With Generalized And Application

Rasha Thnoon Taieb Alrawi
Iraq - Ministry of Education - Kirkuk Open College of Education

corresponding author : Rashaalrawi99@gmail.com

Abstract. *The aim of this paper is to study the Kronecker method then we explain the relationship between proposed method and the matrix. Also we can be generalize this method on the matrix and we give some examples for clear this method and importance in solving a lot of problems.*

Keywords: *Method, Kronecker, Generalized, Application.*

INTRODUCTION:

The Kronecker method is a potent mathematical methodology mostly utilized in matrix theory and linear algebra, and it bears the name of the German mathematician Leopold Kronecker. It entails working with and analyzing matrices, especially when tensor Products are involved [1].

The Kronecker technique is essentially concerned with creating or evaluating big matrices

Through the formation of tensor products of smaller matrices. This method works especially

Well when working with organized matrices or breaking down complicated system into their

More manageable parts. When two matrices, A and B, are Kronecker multiplied, the result is a block matrix called $A \otimes B$, where each element of A is multiplied by the total matrix B[5] . method, which bears the name of the German mathematician Leopold Kronecker. Large In[6]

Matrices may be effectively manipulated by taking use of their tensor product structure, which

Contributes to its flexibility. Letś explore the introduction, generalization, and uses of it.

A mong the most important uses of the Kronecker approach is Numerical analysis, signal processing, linear algebra, and other domains employ the robust

Mathematical methodology known as the Kronecker

1) Overiew of the Kronecker method: [4]

The operation that joins two matrices to create a bigger matrix is called the Kronecker product, represented by the symbol ⊗.

The Kronecker product $\boldsymbol{A} \otimes \boldsymbol{B}$, given two matrices A and B, yields a block matrix by multiplying each element of A by the total matrix B. In formal terms, the Kronecker product is equal to A times $m \times n$ times B.

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* *Rasha Thnoon Taieb Alrawi* Rashaalrawi99@gmail.com

We take this example to clear the Kronecker product

Example 1.1:[4]

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$

So ,the kronecker product $A \otimes B = \begin{bmatrix} 1 \times B & 2 \times B \\ 3 \times B & 4 \times B \end{bmatrix}$

$$= \begin{matrix} 1 \times \\ \\ 3 \times \end{matrix} \begin{bmatrix} 0 & 5 & 0 & 5 \\ 6 & 7 & 6 & 7 \\ 0 & 5 & 0 & 5 \\ 6 & 7 & 6 & 7 \end{bmatrix} \begin{matrix} \times 2 \\ \\ \times 4 \end{matrix} = \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 21 & 24 & 28 & 18 \end{bmatrix}$$

The resulting matrix of size $(2 \times 2) \otimes (2 \times 2) = 4 \times 4 = \boxed{16}$.

Example 1.2 :[17]

$$A = \begin{bmatrix} 1 & -4 & 7 \\ -2 & 3 & 3 \end{bmatrix} , \qquad B = \begin{bmatrix} 8 & -9 & -6 & 5 \\ 1 & -3 & -4 & 7 \\ 2 & 8 & -8 & -3 \\ 1 & 2 & -5 & -1 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} 8 & -9 & -6 & 5 & -32 & 36 & 24 & -20 & 56 & -63 & -42 & 35 \\ 1 & -3 & -4 & 7 & -4 & 12 & 16 & -28 & 7 & -21 & -28 & 49 \\ 2 & 8 & -8 & -3 & -8 & -32 & 32 & 12 & 14 & 56 & -56 & -21 \\ 1 & 2 & -5 & -1 & -4 & -8 & 20 & 4 & 7 & 14 & -35 & -7 \\ -16 & 18 & 12 & -10 & 24 & -27 & -18 & 15 & 24 & -27 & -18 & 15 \\ -2 & 6 & 8 & -14 & 3 & -9 & -12 & 21 & 3 & -9 & -12 & 21 \\ -4 & -16 & 16 & 6 & 6 & 24 & -24 & -9 & 6 & 24 & -24 & -9 \\ -2 & -4 & 10 & 2 & 3 & 6 & -15 & -3 & 3 & 6 & -15 & -3 \end{bmatrix}$$

2) Utilizing the Kronecker method in Applications:

a. Linear Algebra:

The Kronecker product is widely utilized in linear algebra for a number of tasks, including matrix factorizations, eigenvalue and eigenvector computing by giving structured matrices a succinct form.

b. Signal processing:

Filter construction, convolution procedures, and multichannel system analysis all make use of the Kronecker product. It makes it easier to analyze and process by enabling the concise representation of multidimensional signals and systems.

c. Numerical Analysis:

Partial differential equation (PDE) and finite element techniques are two areas in which the Kronecker approach is applied.

If contributes to the reduction of computing complexity by discretizing spatial domains and effectively encoding the resultant matrices.

d. **Quantum Information Theory:**

The Kronecker product is an essential tool for modeling composite quantum systems and their development in both quantum information theory.

In[2], Matrix work on Kronecker product occasionally treats matrices as vectors and occasionally

Makes vectors into matrices. We employ the vec operation to be exact about these reshaping's.

If $X \in R^{m \times n}$ so, the $nm \times 1$ vector vec(X) is produced by stacking the columns of X. It is easy to prove the following equivalency if C, X, and B are matrices and the product CXB^T is defined: $y = CXB^T \equiv y = (B \otimes C)\, X$

Where $x = vec(X)$ and $y = vec(Y)$

The Kronecker product, like every significant mathematical operation, has undergone modifications and specializations to accommodate intriguing new applications.

A theory of Array algebra is presented by Rauhala [7] and is applicable to certain photogrammetric issues. Also [10].

Regalia and mitral [8] have described a variety of rapid unitary transforms

Using the Kronecker product and its numerous generalizations.

$$\{\boldsymbol{A_1}, \ldots \mathbf{A_m}\} \otimes \mathbf{B} = \mathbf{A_1} \otimes \mathbf{B(1,;)} \ \mathbf{A_2} \otimes \mathbf{B(2,;)} \ldots \boldsymbol{A_m} \otimes \mathbf{B(m,;)}$$

Asample generalization that appears in their presentation is that B has m rows and A_1,...,A_m are supplied matrices of the same size.

In[9], a strong Kronecker product is introduced as an additional generalization that facilitates the analysis of specific orthogonal matrix multiplication issues. An $m \times n$ block matrix $A = (Aij)$ is the strong Kronecker product of a p $\times n$ block matrix $C = (Cij)$ and a $m \times p$ block matrix $B = (Bij)$,

where: $\boldsymbol{A_{ij} = B_{i1} \otimes C_{1j} + \ldots + B_{ip} \otimes C_{pj}}$

3) Definition of generalized Kronecker product :

Definition 3.1: [3]

The Kronecker product of a matrix A by $m \times n$ and a matrix B by $p \times q$ in algebra system

$(s, \circ, +)$ can be denoted by $A \otimes B$, which is defined as follows .

$$A \otimes B = [a_{ij} \circ \mathrm{B}] = \begin{bmatrix} a_{11} \circ B & a_{21} \circ B \dots & a_{1n} \circ B \\ a_{21} \circ B & a_{22} B \dots & a_{21} \circ B \\ a_{m1} \circ B & a_{m2} \dots & a_{mn} \circ B \end{bmatrix} \in F^{(mp)\times(nq)}$$

Definition 3.2 : [4]

If A is an $m \times n$ and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $pm \times qn$ block matrix .

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n} \circ B \\ \vdots & \ddots & \vdots \\ a_{m1} \circ B & \dots & a_{mn} \circ B \end{bmatrix}$$

more explicitly

$$A \otimes B = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \dots & a_{11}b_{1q} & \dots & \dots & a_{1n}b_{11} & a_{1n}b_{12} & \dots & a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \dots & a_{11}b_{2q} & \dots & \dots & a_{1n}b_{21} & a_{1n}b_{22} & \dots & a_{1n}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \dots & a_{11}b_{pq} & \dots & \dots & a_{1n}b_{p1} & a_{1n}b_{p2} & \dots & a_{1n}b_{pq} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \dots & a_{m1}b_{1q} & \dots & \dots & a_{mn}b_{11} & a_{mn}b_{12} & \dots & a_{mn}b_{pq} \\ a_{m1}b_{21} & a_{m1}b_{22} & \dots & a_{m1}b_{2q} & \dots & \dots & a_{mn}b_{21} & a_{mn}b_{22} & \dots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}b_{21} & a_{m1}b_{22} & \dots & a_{m1}b_{2q} & \dots & \dots & a_{mn}b_{21} & a_{mn}b_{22} & \dots & a_{mn}b_{2q} \end{bmatrix}$$

4) Properties of generalized Kronecker product and some examples :

Properties 4.1:[3]

For $A_{m\times n}$ and $B_{p\times q}$ generally

1) $\boldsymbol{A \otimes B \neq B \otimes A}$
2) The Kronecker product of arbitrary matrix and zero matrix equal zero

i.e. $\boldsymbol{A \otimes 0 = 0 \otimes A = 0}$ **.**

3) If $(S, \circ)$ is a **commutative semi-group**, and for arbitrary α and β ,

$$(\boldsymbol{\alpha} \circ \boldsymbol{A}) \otimes (\boldsymbol{\beta} \circ \boldsymbol{B}) = (\boldsymbol{\alpha} \circ \boldsymbol{\beta}) \circ (\boldsymbol{A} \circ \boldsymbol{B}) .$$

4) If $(s, \circ)$ is a **commutative semi-group** and for $A_{m\times n}$,$B_{n\times k}$,$C_{t\times p}$ and $D_{p\times q}$, $(A \circ B) \otimes (C \circ D)$.
5) If $(s, \circ, +)$ is a ring , and for $A_{m\times n}$,$B_{p\times q}$

$$A\otimes (B \pm C) = (A\otimes B) \pm (A\otimes C)$$
$$(B \pm C)\otimes A = (B\otimes A) \pm (C\otimes A)$$

6) If $(s,\circ)$ is a **commutative algebraic system**, and for $A_{m\times n}$ and $B_{p\times q}$

$$(\boldsymbol{A}\otimes\boldsymbol{B})^{\boldsymbol{T}} = \boldsymbol{A}^{\boldsymbol{T}}\otimes\boldsymbol{B}^{\boldsymbol{T}}$$

7) If$(s,+,\circ)$ is a ring and for $A_{m\times n}$,$B_{m\times n}$,$C_{p\times q}$ and $D_{p\times q}$

$$(\boldsymbol{A}+\boldsymbol{B})\otimes(\boldsymbol{C}+\boldsymbol{D}) = \boldsymbol{A}\otimes\boldsymbol{C} + \boldsymbol{A}\otimes\boldsymbol{D} + \boldsymbol{B}\otimes\boldsymbol{C} + \boldsymbol{B}\otimes\boldsymbol{D}\,.$$

8) If $(s,\circ)$ is a **commutative semi-group, and for**$\boldsymbol{A}_{\boldsymbol{m}\times\boldsymbol{n}}$, $\boldsymbol{B}_{\boldsymbol{k}\times\boldsymbol{l}}$, $\boldsymbol{C}_{\boldsymbol{p}\times\boldsymbol{q}}$ **and** $\boldsymbol{D}_{\boldsymbol{r}\times\boldsymbol{s}}$

$$,\ (\boldsymbol{A}\otimes\boldsymbol{B})\otimes(\boldsymbol{C}\otimes\boldsymbol{D})$$

9) If $(s,\circ)$ is a **semi-group, and for**$\boldsymbol{A}_{\boldsymbol{m}\times\boldsymbol{n}}$, $\boldsymbol{B}_{\boldsymbol{k}\times\boldsymbol{l}}$, $\boldsymbol{C}_{\boldsymbol{p}\times\boldsymbol{q}}$

$$\boldsymbol{A}\otimes(\boldsymbol{B}\otimes\boldsymbol{C}) = (\boldsymbol{A}\otimes\boldsymbol{B})\otimes\boldsymbol{C}$$

Kronecker product has many interesting properties and application, and this example demonstrates how it combines elements of two matrices to form a larger block matrix .

Examples 4.2: [4]

If A and B both are 2 × 2 partitioned matrices

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22}\end{bmatrix} = \begin{bmatrix}7 & 4 & 1\\ 8 & 5 & 2\\ 9 & 6 & 3\end{bmatrix}$$

$$\textbf{We get } \boldsymbol{A}\circ\boldsymbol{B} = \begin{bmatrix}\boldsymbol{A}_{11}\circ\boldsymbol{B} & \boldsymbol{A}_{12}\circ\boldsymbol{B}\\ \boldsymbol{A}_{21}\circ\boldsymbol{B} & \boldsymbol{A}_{22}\circ\boldsymbol{B}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2 & 4 & 7 & 8 & 14 & 3 & 12 & 21\\ 4 & 5 & 16 & 28 & 20 & 35 & 6 & 24 & 42\\ 3 & 6 & 6 & 9 & 12 & 18 & 9 & 18 & 9\\ 8 & 10 & 20 & 32 & 25 & 40 & 12 & 30 & 48\\ 12 & 15 & 24 & 36 & 30 & 45 & 18 & 36 & 54\\ 7 & 8 & 28 & 49 & 32 & 56 & 9 & 36 & 63\\ 14 & 16 & 35 & 56 & 40 & 64 & 18 & 45 & 72\\ 21 & 24 & 42 & 63 & 48 & 72 & 27 & 54 & 81\end{bmatrix}$$

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